PP36784. Proposed by Michaly Bencze.

Determine all $x \in \left[0, \frac{\pi}{2}\right]$ for which $4^{\sin x} + 4^{\cos x} \le 2^{1+\sqrt{2}}$.

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Let $f(x) := 4^{\sin x} + 4^{\cos x}$. Since $f(x) = f\left(\frac{\pi}{2} - x\right)$ then $\max\left\{f(x) \mid x \in \left[0, \frac{\pi}{2}\right]\right\} =$ $\max\left\{f(x) \mid x \in \left[0, \frac{\pi}{4}\right]\right\}$. We have $f'(x) = \ln 4(4^{\sin x}\cos x - 4^{\cos x}\sin x)$.

Noting that $f'(0) = \ln 4$ we assume for further that $x \in (0, \pi/4]$.

Then $f'(x) = \dfrac{\ln 4}{\sin x \cdot \cos x}\left(\dfrac{4^{\sin x}}{\sin x} - \dfrac{4^{\cos x}}{\cos x}\right)$ and we will explore sign of $\dfrac{4^{\sin x}}{\sin x} - \dfrac{4^{\cos x}}{\cos x}$.

Let $h(t) := \dfrac{4^t}{t}$, where $t \in \left(0, 1/\sqrt{2}\,\right]$. Noting that $0 < \sin x < \cos x < 1/\sqrt{2}$ for any $x \in (0, \pi/4)$ we obtain by Mean Value Theorem that

$\dfrac{h(\cos x) - h(\sin x)}{\cos x - \sin x} = h'(c)$, where $c \in (\sin x, \cos x)$ and $h'(t) = \dfrac{4^t(t\ln 4 - 1)}{t^2}$.

Since $c < 1/\sqrt{2}$ then $h'(c) = \dfrac{4^c(c\ln 4 - 1)}{c^2} < \dfrac{4^c\left(\frac{1}{\sqrt{2}}\ln 4 - 1\right)}{c^2} = \dfrac{4^c\left(\ln 4 - \sqrt{2}\right)}{\sqrt{2}\,c^2} < 0$

because* $\ln 4 < \sqrt{2}$. Hence, $h(\cos x) - h(\sin x) < 0 \Leftrightarrow \dfrac{4^{\sin x}}{\sin x} - \dfrac{4^{\cos x}}{\cos x} > 0$

and, therefore, $f'(x) > 0$ for any $x \in (0, \pi/4]$. Since $f(x)$ increase on $(0, \pi/4]$ then $f(x) \le f\left(\frac{\pi}{4}\right) = 2^{1+\sqrt{2}} \Leftrightarrow 4^{\sin x} + 4^{\cos x} \le 2^{1+\sqrt{2}}$ for any $x \in (0, \pi/4]$

and $\max\left\{f(x) \mid x \in \left[0, \frac{\pi}{2}\right]\right\} = 2^{1+\sqrt{2}}$.

* We will prove without using softs that $\ln 4 < \sqrt{2}$.

Indeed, $\frac{7}{5} < \sqrt{2} \Leftrightarrow 49 < 50$ and $\ln 4 < \frac{7}{5} \Leftrightarrow 4^5 < e^7 \Leftrightarrow 2^{10} < e^7$ where latter inequality holds because $e^7 > 2.7^9$ and $2.7^9 > 2^{10}$ (since $2.7^9 = \dfrac{3^{27}}{10^9} > \dfrac{3^{27}}{2^{30}}$ and $\dfrac{3^{27}}{2^{30}} > 2^{10} \Leftrightarrow 3^{27} > 2^{40} \Leftrightarrow \left(\frac{3}{2}\right)^{27} > 2^{13} \Leftrightarrow \frac{3}{2} \cdot \left(\frac{9}{4}\right)^{13} > 2^{13} \Leftrightarrow \frac{3}{2} > \left(\frac{8}{9}\right)^{13}$).